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Vibration Analysis of Mistuned Bladed-Disk Assemblies—Inverse Approach

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This paper deals with the development of a general analysis method that aims at solving the inverse response problem where some of the key structural parameters are determined from knowledge of the system's response. This proposed method of analysis was applied to the specific case of a 12-bladed disk to determine the range of individual blade cantilever frequency scatter so that all blade responses lay below an allowable maximum level. This scatter or manufacturing tolerance was computed statistically, and the analysis was verified using a direct Monte Carlo simulation. The relationship between the allowable response increase due to mistuning and the corresponding manufacturing tolerances was also investigated, and the presence of a critical mistuning threshold was noted. The existence and the nonuniqueness of the inverse solution is discussed in some detail. A number of numerical problems are identified, and routes for their avoidance are suggested. Finally, a comparison of the direct and inverse analysis methods for bladed-disk mistuning studies is presented and the suitability of the inverse method to bladed-disk design is discussed.

Nomenclature

$[C]$ = viscous damping matrix
c = viscous damping coefficient
$[D]$ = structural damping matrix
$\{\hat{f}\}$ = amplitude of force vector
$\bar{G}$ = mean of blade first cantilever frequencies
G_j = first cantilever frequency of jth blade
g_i = ith functional relation
$[K]$ = stiffness matrix
K_d = sectorial disk stiffness
k_g = grounding stiffness
k^* = complex blade stiffness
$[M]$ = mass matrix
M_d = sectorial disk mass
m = blade mass, number of degrees of freedom
N = number of blades
ND = nodal diameter
n = number of functional relations or equations
P = cumulative density function (cdf)
$\{\hat{q}\}$ = amplitude of response vector
$\{R\}$ = residual vector
r = engine order of excitation
$[S]$ = sensitivity matrix
s = number of excitation frequencies
$\{U_j\}$ = complex response of jth sector
u = number of unknown structural parameters
v = total number of unknowns
w = number of coordinates at which the complex response levels are known
X_j = response amplitude of jth blade
${}_IX_j$ = imaginary part of jth blade's response amplitude
${}_MX_j$ = magnitude of jth blade's response amplitude
${}_RX_j$ = real part of jth blade's response amplitude
x_j = response of jth blade
Y_j = response amplitude of jth disk sector
${}_IY_j$ = imaginary part of jth disk sector's response amplitude
${}_RY_j$ = real part of jth disk sector's response amplitude
y_j = response of jth disk sector
$\{\gamma\}$ = vector of unknowns
γ_i = element of $\{\gamma\}$
$\{\Delta\gamma\}$ = correction vector at each iteration
ζ = viscous damping ratio
η = hysteretic damping ratio
θ_r = interblade phase angle ($2\pi r/N$)

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I. Introduction

TRADITIONAL vibration analyses of mistuned bladed-disks are usually concerned with dynamic response predictions of assemblies with predetermined configurations.[1-4] The main problem addressed by such studies is the determination of the actual response due to a given amount of mistuning. In other words, the response is determined as a function of mistuning, the latter being the independent variable. Almost all mistuning studies reported in the literature fall in this category since they address the direct problem of finding the consequences of mistuning. However, from a design point of view, what is of real interest is the knowledge of maximum acceptable degree of mistuning (or manufacturing tolerance) for which the response increase with respect to the tuned system will not exceed an allowable limit. In other words, the key requirement is the establishment of a direct relationship which gives the mistuning scatter for a specified allowable maximum response. The input to this problem is, therefore, the maximum allowable response when the solution consists of specific mistuning patterns, a situation which is precisely the reverse of traditional mistuning studies which aim at predicting response increases due to given mistuning patterns. It is, therefore, appropriate to classify this type of investigation as that of an inverse problem.

From the outset, it must be stressed that the determination of a solution for an inverse problem is far more difficult than that for a direct problem because inverse solutions are usually sought iteratively by analyzing a very large number of systems since analytical formulations cannot normally be found. However, it is possible to simplify the task by using statistical tools, in which case the number of systems to be analyzed can be reduced dramatically. This is the approach adopted in this study. In any case, an inverse problem may not have a solution at all, or the solution found may be one of many such solutions, all satisfying the imposed constraints.[5]

The main purpose of this paper is to develop directly an inverse method that relates the maximum allowable response increase to the corresponding scatter in individual blade properties. This scatter is defined by manufacturing tolerances and other blade material properties, and the current engineering practice is to base its choice on past experience rather than any rigorous analysis. This inevitably leads to large response increases over the tuned case, and almost all current mistuning analyses focus on the consequences of this inherent bladed-disk mistuning. However, there is an alternative approach which is to start from the maximum allowable response increase and to determine the manufacturing tolerances so that the response stays below this maximum increase throughout the frequency range of interest. In this case, a direct relationship between the response increase and the amount of mistuning that causes it becomes of considerable interest to turbomachinery designers and the main purpose of this paper is to develop such a methodology as an extension of Refs. 6 and 7. If adopted, this route is expected to lower production costs since the manufacturing tolerances will be kept tight when needed and relaxed when appropriate.

The mistuning problem is further compounded by the fact that a certain amount of mistuning is considered to be desirable since this feature is believed to increase flutter stability margins.[8] A variant of the inverse approach also can be used for passive flutter control by specifying both the maximum allowable response increase and the mistuning pattern which is most likely to minimize the flutter response. However, the present study is confined to blades with structural coupling only since the addition of the aerodynamic coupling can change the mistuned response significantly.

II. Theory

A. Mathematical Formulation

The equations of motion for a linear system with m degrees of freedom can be written as

$$([K + iD] - \omega^2[M] + i\omega[C])_{m\times m}\{\hat{q}\}_{m\times 1} - \{\hat{f}\}_{m\times 1} = 0 \quad (1)$$

where all symbols used are defined in the Nomenclature.

The matrix equation (1) can be separated into its real and imaginary parts, thus giving $2m$ real equations. These $2m$ real equations can be written explicitly at s excitation frequencies to obtain $n = 2sm$ nonlinear functional relations of the form

$$g_i(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_v) = 0 \qquad i = 1, 2, 3, \ldots, n \quad (2)$$

where $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_v$ are the v unknown structural and response parameters. Here $v = u + n$ with u the total number of unknown structural parameters and n the total number of unknown response levels. Also, g is a known nonlinear function of $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_v$.

The simultaneous solution of n functional relations obtained from the equations of motion cannot yield all v unknown parameters since there are $(v = u + n)$ unknowns for n equations. However, it is possible to solve this underdetermined problem by introducing additional constraints: if the response levels are known for w coordinates at s excitation frequencies, then the number of unknown response levels will be reduced from $n = 2sm$ to $n = 2sm - 2sw$. Therefore, the total number of unknowns will be $v = u + 2s(m - w)$.

It is also possible to make the problem overdetermined by choosing w such that $2sw > u$. In this case, a solution can be obtained via a weighted least-squares approach or a singular value decomposition for the u unknown structural parameters and the $2s(m - w)$ unknown response levels.

As mentioned before, the n functional relations g_i for the v unknowns γ_j are not linear and, hence, it is appropriate to employ an iterative solution technique.[9] The Newton-Raphson algorithm was used in this work because of its rapid convergence properties when there is a root within the search interval.

Let the vector $\{\gamma\}$ contain the initial estimates of the v unknowns γ_j. Each of the functions g_i can be expanded in a Taylor series in the neighborhood of vector $\{\gamma\}$ as

$$g_i(\{\gamma\} + \{\Delta\gamma\}) = g_i(\{\gamma\}) + \sum_{j=1}^{v} \frac{\partial g_i}{\partial \gamma_j}\delta\gamma_j + \mathcal{O}(\{\Delta\gamma\}^2) \quad (3)$$

where $\{\Delta\gamma\}$ can be considered as a correction vector for the initial estimates. Equation (3) can be linearized by neglecting second- and higher-order terms, and a solution can be obtained for $\{\Delta\gamma\}$ which aims at nullifying all functions g_i simultaneously. This can be written in matrix form as

$$[S]_{n\times v}\{\Delta\gamma\}_{v\times 1} = -\{R\}_{n\times 1} \quad (4)$$

where

$$[S] = \begin{bmatrix} \frac{\partial g_1}{\partial \gamma_1} & \frac{\partial g_1}{\partial \gamma_2} & \frac{\partial g_1}{\partial \gamma_3} & \cdots & \frac{\partial g_1}{\partial \gamma_v} \\ \frac{\partial g_2}{\partial \gamma_1} & \frac{\partial g_2}{\partial \gamma_2} & \frac{\partial g_2}{\partial \gamma_3} & \cdots & \frac{\partial g_2}{\partial \gamma_v} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \frac{\partial g_n}{\partial \gamma_1} & \frac{\partial g_n}{\partial \gamma_2} & \frac{\partial g_n}{\partial \gamma_3} & \cdots & \frac{\partial g_n}{\partial \gamma_v} \end{bmatrix}$$

$$\{\Delta\gamma\} = \begin{Bmatrix} \Delta\gamma_1 \\ \Delta\gamma_2 \\ \Delta\gamma_3 \\ \cdot \\ \cdot \\ \cdot \\ \Delta\gamma_v \end{Bmatrix}; \qquad \{R\} = \begin{Bmatrix} g_1\{\gamma\} \\ g_2\{\gamma\} \\ \cdot \\ \cdot \\ g_n\{\gamma\} \end{Bmatrix}$$

The sensitivity matrix $[S]$ and the residual vector $\{R\}$ are formed using initial (and then corrected) values of vector $\{\gamma\}$. In other words, Eq. (4) is solved for $\{\Delta\gamma\}$ in an iterative fashion

$$\{\gamma\}^{\text{new}} = \{\gamma\}^{\text{old}} + \{\Delta\gamma\} \quad (5)$$

until convergence is achieved.

B. Application to a Bladed Disk

It is proposed to represent the bladed-disk assembly using the lumped parameter model[10] of Fig. 1. Previous studies[11,12] indicate that such simple models are able to reproduce the basic vibration characteristics of bladed-disk assemblies with remarkable affinity, and hence they can be used for qualitative analyses of turbomachinery blades in spite of their marked simplicity. In the case of this particular study, the scatter in individual blade properties was simulated by allowing the first cantilever frequency of each blade, defined by parameter G_j, to be an unknown variable.

Referring to Fig. 1 where a typical bladed-disk sector is defined using two degrees of freedom, x for the blade mass and y for the corresponding disk section, the equations of motion can be written as

$$\ddot{x}_j + 2\zeta G_j\dot{x}_j + G_j^2(1 + i\eta)(x_j - y_j) - [f_j(t)/m] = 0$$

$$\frac{M_d}{m}\ddot{y}_j + G_j^2(1 + i\eta)(y_j - x_j) + \frac{k_g}{m}y_j + \frac{K_d}{m}(2y_j - y_{j+1} - y_{j-1}) = 0 \quad (6)$$

where $f_j(t)$ is the external force applied at the jth blade, and the remaining symbols are defined in the Nomenclature. Assuming simple harmonic motion

$$x_j = X_j e^{i(\omega t)} \tag{7a}$$

$$y_j = Y_j e^{i(\omega t)} \tag{7b}$$

and inserting Eqs. (7a) and (7b) into Eq. (6) gives

$$\begin{aligned} &-\omega^2 X_j + i2\zeta G_j \omega X_j + G_j^2(1 + i\eta)(X_j - Y_j) \\ &\quad - (F_j/m) = 0 \\ &-\omega^2 \frac{M_d}{m} Y_j + G_j^2(1 + i\eta)(Y_j - X_j) + \frac{k_g}{m} Y_j \\ &\quad + \frac{K_d}{m}(2Y_j - Y_{j+1} - Y_{j-1}) = 0 \end{aligned} \tag{8}$$

In bladed-disk applications, a specific type of external forcing, namely, engine-order excitation, is of particular interest. In this case the external force F_j can be written as

$$F_j = F_0 e^{i[(j-1)\theta_r]} \tag{9}$$

where F_0 is the amplitude of the force vector, $\theta_r = 2\pi r/N$ is the interblade phase angle, r is the order of the engine excitation, and N is the total number of blades.

The response amplitudes of the jth blade and the corresponding disk sector at any exciting frequency ω_ℓ can be separated into its real and imaginary parts as

$$\begin{aligned} X_j(\omega_\ell) &= {}_R X_j(\omega_\ell) + i\,{}_I X_j(\omega_\ell) \\ Y_j(\omega_\ell) &= {}_R Y_j(\omega_\ell) + i\,{}_I Y_j(\omega_\ell) \end{aligned} \tag{10}$$

Substituting Eq. (10) into Eq. (8) gives four functional relations for each bladed-disk sector at exciting frequency ω_ℓ

$$\begin{aligned} g_{5s(j-1)+5\ell-4} &= -\omega_\ell^2\,{}_R X_j(\omega_\ell) - 2\zeta G_j \omega_\ell\,{}_I X_j(\omega_\ell) \\ &\quad + G_j^2[{}_R X_j(\omega_\ell) - \eta\,{}_I X_j(\omega_\ell) - {}_R Y_j(\omega_\ell) + \eta\,{}_I Y_j(\omega_\ell)] \\ &\quad - \frac{F_0}{m}\cos\left(\frac{2\pi(j-1)r}{N}\right) = 0 \\ g_{5s(j-1)+5\ell-3} &= -\omega_\ell^2\,{}_I X_j(\omega_\ell) + 2\zeta G_j \omega_\ell\,{}_R X_j(\omega_\ell) \\ &\quad + G_j^2[{}_I X_j(\omega_\ell) + \eta\,{}_R X_j(\omega_\ell) - {}_I Y_j(\omega_\ell) - \eta\,{}_R Y_j(\omega_\ell)] \\ &\quad - \frac{F_0}{m}\sin\left(\frac{2\pi(j-1)r}{N}\right) = 0 \\ g_{5s(j-1)+5\ell-2} &= -\omega_\ell^2 \frac{M_d}{m}\,{}_R Y_j(\omega_\ell) + G_j^2[{}_R Y_j(\omega_\ell) \\ &\quad - \eta\,{}_I Y_j(\omega_\ell) - {}_R X_j(\omega_\ell) + \eta\,{}_I X_j(\omega_\ell)] + \frac{k_g}{m}\,{}_R Y_j(\omega_\ell) \\ &\quad + \frac{K_d}{m}[2\,{}_R Y_j(\omega_\ell) - {}_R Y_{j+1}(\omega_\ell) - {}_R Y_{j-1}(\omega_\ell)] = 0 \\ g_{5s(j+1)+5\ell-1} &= -\omega_\ell^2 \frac{M_d}{m}\,{}_I Y_j(\omega_\ell) + G_j^2[{}_I Y_j(\omega_\ell) \\ &\quad + \eta\,{}_R Y_j(\omega_\ell) - {}_I X_j(\omega_\ell) - \eta\,{}_R X_j(\omega_\ell)] + \frac{k_g}{m}\,{}_I Y_j(\omega_\ell) \\ &\quad + \frac{K_d}{m}[2\,{}_I Y_j(\omega_\ell) - {}_I Y_{j+1}(\omega_\ell) - {}_I Y_{j-1}(\omega_\ell)] = 0 \\ &\ell = 1, 2, 3, \ldots, s \quad \text{and} \quad j = 1, 2, 3, \ldots, N \end{aligned} \tag{11}$$

Table 1 Bladed-disk structural parameters

$N = 12$	$m = 0.05$ kg	$M_d = 0.18$ kg
$k^* = 65384(1 + i0.002)$ N/m	$K_d = 9411531$ N/m	$k_g = 63$ N/m
$(\bar{G}/2\pi) = 182$ Hz	$c = 1.144$ Ns²/m	

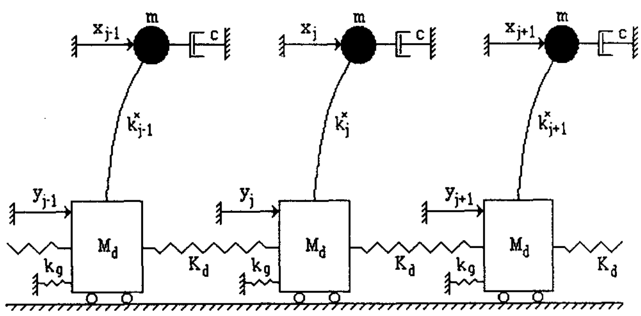

Fig. 1 Lumped parameter model of mistuned bladed-disk assembly.

where the subscript of g refers to equation numbering. The unknowns in the set of equations (11) are the complex response levels X_j and Y_j, and the cantilever frequency of each blade G_j, a feature which makes the problem both nonlinear and underdetermined since there are $4sN$ equations for $4sN + N$ unknowns. Therefore, additional information is needed to solve the problem. However, no explicit data are available for the actual values of the real and imaginary parts of the response levels, the only information at hand being the magnitude of the maximum allowable response for the jth blade at excitation frequency ω_ℓ: ${}_M X_j(\omega_\ell)$. Using this additional information as a constraint equation at s excitation frequencies yields

$$\begin{aligned} g_{5s(j-1)+5\ell} &= {}_R X_j^2(\omega_\ell) + {}_I X_j^2(\omega_\ell) - {}_M X_j^2(\omega_\ell) = 0 \\ \ell &= 1, 2, 3, \ldots, s \quad \text{and} \quad j = 1, 2, 3, \ldots, N \end{aligned} \tag{12}$$

Equations (11) and (12) now constitute an overdetermined set of $5sN$ nonlinear equations for $(4s + 1)N$ unknowns, the unknowns being $4sN$ complex responses and cantilever frequencies for N blades. Equations (11) and (12) can now be put in a matrix form as

$$[S]_{5sN\times(4s+1)N}\{\Delta\gamma\}_{(4s+1)N\times 1} = -\{R\}_{5sN\times 1} \tag{13}$$

where

$$\{\Delta\gamma\}^T = \{\Delta G_1, \{\Delta U_1\}, \Delta G_2, \{\Delta U_2\}, G_3, \{U_3\}, \ldots, \Delta G_N, \{\Delta U_N\}\}$$

$$\{R\}^T = \{g_1(\{\gamma\}), g_2(\{\gamma\}), g_3(\{\gamma\}), \ldots, g_{5sN}(\{\gamma\})\}$$

$$\begin{aligned} \{\Delta U_j\}^T = \{&\Delta_R X_j(\omega_1), \Delta_I X_j(\omega_1), \Delta_R Y_j(\omega_1), \Delta_I Y_j(\omega_1), \ldots, \\ &\Delta_R X_j(\omega_\ell), \Delta_I X_j(\omega_\ell), \Delta_R Y_j(\omega_\ell), \Delta_I Y_j(\omega_\ell), \ldots, \\ &\Delta_R X_j(\omega_s), \Delta_I X_j(\omega_s), \Delta_R Y_j(\omega_s), \Delta_I Y_j(\omega_s)\} \end{aligned}$$

and the explicit formulation of the $[S]$ matrix is given in the Appendix.

Equation (13) can now be solved iteratively by using the technique described in the previous section.

III. Case Studies

A. Introduction

Except where stated otherwise, the structural parameters shown in Table 1 were used in this investigation. These values were determined for a particular bladed disk using a semiempirical method described in Ref. 11. Approximate damping values were also included in the model.

One of the main difficulties encountered when analyzing the dynamic behavior of mistuned bladed-disk assemblies is the dependence of the results on specific configurations; in other words, an N-bladed assembly possesses $N!$ maximum response envelopes since N blades can be arranged in $N!$ ways. The problem is further complicated by the fact that the response is also dependent on the number of blades, the engine-order of excitation, and the excitation frequency. The correct choice of these critical parameters is, therefore, of crucial importance to the validity of the conclusions drawn from the numerical studies. In this particular study, it is proposed to use some of the main findings of Refs. 6 and 7 which will be summarized here.

1) The combined effect of changing the total number of blades N and the engine-order of excitation r can be obtained by considering a single parameter, namely, the interblade phase angle $\theta_r = 2\pi r/N$. Hence, the dynamic behavior of large bladed-disk systems can be deduced from that of much smaller ones, a feature which can save a very substantial amount of CPU time. It was, therefore, decided to use 12 blades in this particular study.

2) Previous work, reported in Ref. 7, showed that the maximum resonant response increase due to mistuning occurred for an interblade phase angle of about 60 deg in the case of this particular bladed-disk configuration. Since it was decided to use 12 blades, the corresponding engine order of excitation was found to be $60 \times 12/360 = 2$ from the relationship given in paragraph 1.

3) Statistical analysis of simulated response data from hundreds of mistuned bladed-disk assemblies indicated that the most responsive blades reached their maximum amplitude in the vicinity of the corresponding tuned system's natural frequency.[6] This finding was used in this study to define the excitation frequency at which maximum allowable response levels were defined.

B. Selection of Blade Response Levels for the Inverse Solution

The magnitudes of all of the mistuned blade responses were chosen in a semirandom manner for a bladed disk vibrating in its mistuned two-nodal diameter mode. The response curve corresponding to the tuned case is shown in Fig. 2 where the maximum, tuned and minimum blade responses are indicated by points A, B, and C. In this particular case, point A corresponds to a maximum allowable response increase of 30%, point B represents the tuned system response, and point C defines the lowest blade response at that frequency, the determination of which will be explained later. The sampling of the response levels has already been termed semirandom because out of a total of 12 blade responses, at least one and at most three were forced to lie between points A and B, thus simulating a typical case where only a few blades experience higher response levels than those found in the corresponding tuned system. Furthermore, the highest response was automatically set to an increase of 30% to ensure that every mistuned system had at least one blade experiencing the maximum allowable response increase. The response levels of the remaining blades were selected randomly between points B and C.

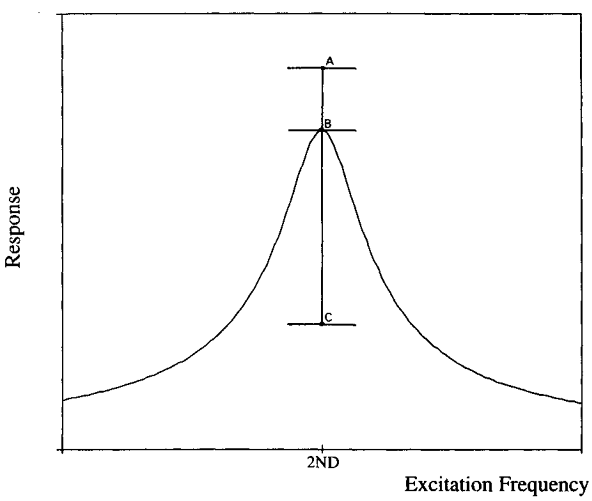

Fig. 2 **Maximum (point A), tuned (point B), and minimum (point C) blade responses (12-bladed disk around second mode).**

The next difficulty lies in the determination of a suitable value for the lower response bound defined by point C. It is reasonable to assume that the upper and lower bounds for mistuned blade response levels will converge toward the same value with decreasing mistuning since, in the limit, all responses will pass through point B. Conversely, the response at point C will decrease with increasing allowable response increase. By making use of this information, the lower bound was determined by trial and error as follows: a lower bound close to the tuned response at point B was chosen initially, and its value was decreased until about 70% of cases with all 12 blade responses sampled between lower and upper bounds yielded solutions.

C. Verification of the Results

The validation of the results obtained from the inverse solution is straightforward since the blade cantilever frequencies obtained from the inverse solution can be used to form a spatial model of the assembly and the response levels can be computed directly. Table 2 lists the results from such a procedure, and the relative difference between the assumed and recovered response levels is less than 0.5% in all cases.

D. Statistical Analysis

The solution technique developed in the previous section can now be applied to the set of 12 blade responses to find the corresponding mistuned system. However, even if such a mistuned system can be found, the response levels of some of its blades may well exceed the allowable response increase at excitation frequencies other than those specified for selecting the blade responses. A typical example for such a situation is illustrated in Fig. 3 where the response levels of all 12 blades are shown against the normalized excitation frequency. All responses are normalized with respect to corresponding tuned resonant response, and the excitation frequency is normalized to the two-nodal diameter natural frequency of the tuned system; it is immediately seen that there is one blade whose response level exceeds the 30% limit for values of the normalized excitation frequency between 1.0 and 1.1. Perhaps more importantly, there is no guarantee that the situation will remain the same when the same set of blades are rearranged around the disk. All of these observations indicate that a statistical approach with a large enough sample size must be used to address the inverse problem.

Mistuning configurations, characterized by individual blade cantilever frequencies, were determined for 500 bladed disks, each having one blade experiencing a maximum of 30% response increase. The maximum (G_{max}) and the minimum (G_{min}) blade frequencies for all 500 assemblies are plotted in Fig. 4 by normalizing all values with respect to the tuned blade cantilever frequency ($\bar{G}$). The same information is conveyed in a different format in Fig. 5 where the percentage tolerance, defined as $100 \times (G_{max} - G_{min})/2\bar{G}$, is plotted for the same 500 bladed disks. If the worst case is considered, it is seen that a tolerance of as little as 0.73% can cause a 30% response increase, suggesting that manufacturing tolerances for individual blades should be set such that the resulting cantilever frequencies are within (1 ± 0.0073) of their nominal value.

The findings from the inverse solution were also checked statistically by using a direct approach where unknown response levels were computed for a given mistuning percentage obtained from the inverse solution. The individual blade can-

Table 2 Verification of the inverse solution: mistuned response is normalized with respect to tuned response level and the blade cantilever frequency is normalized with respect to the tuned blade cantilever frequency

	Inverse solution		Direct solution		
Blade no.	Input blade res. (normalized)	Output blade freq. (normalized)	Input blade freq. (normalized)	Output blade res. (normalized)	Differ., %
1	0.7857	1.0133	1.0133	0.7830	0.34
2	0.9143	0.9942	0.9942	0.9138	0.05
3	1.1766	0.9952	0.9952	1.1761	0.04
4	0.7220	1.0214	1.0214	0.7214	0.08
5	0.7721	1.0129	1.0129	0.7722	−0.01
6	1.0230	0.9942	0.9942	1.0220	0.10
7	0.7830	1.0086	1.0086	0.7828	0.02
8	0.9804	1.0042	1.0042	0.9792	0.12
9	0.9243	0.9973	0.9973	0.9227	0.17
10	1.3000	0.9853	0.9853	1.2954	0.36
11	0.8393	1.0458	1.0458	0.8433	−0.47
12	0.7204	1.0356	1.0356	0.7220	−0.22

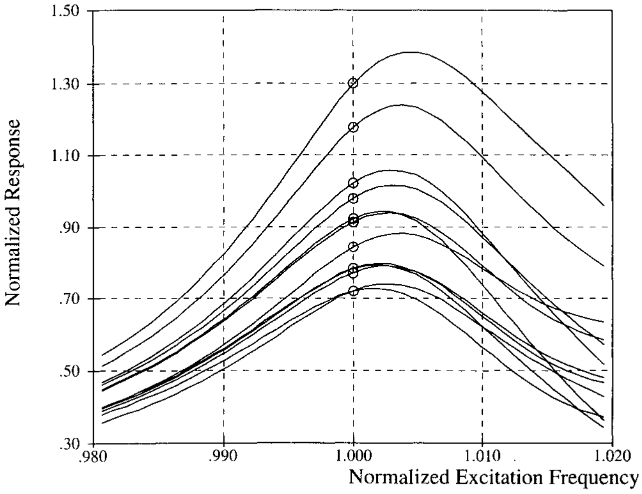

Fig. 3 Inadequacy of relying on individual excitation frequencies for defining the allowable maximum response increase.

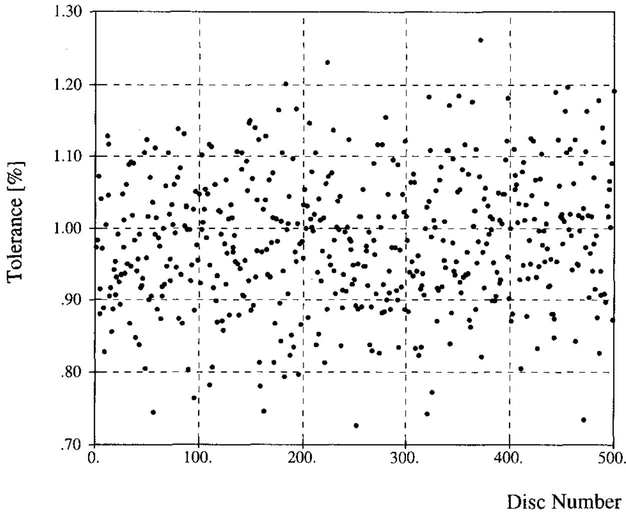

Fig. 5 Maximum relative deviation for 500 bladed disks.

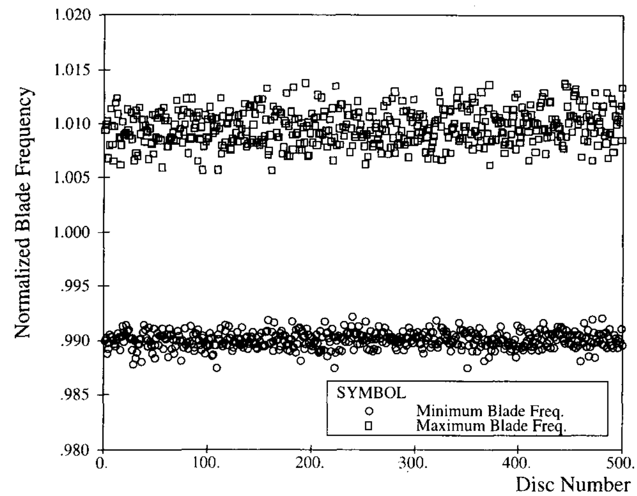

Fig. 4 The minimum and the maximum blade frequencies for 500 bladed disks which satisfy 30% response increase.

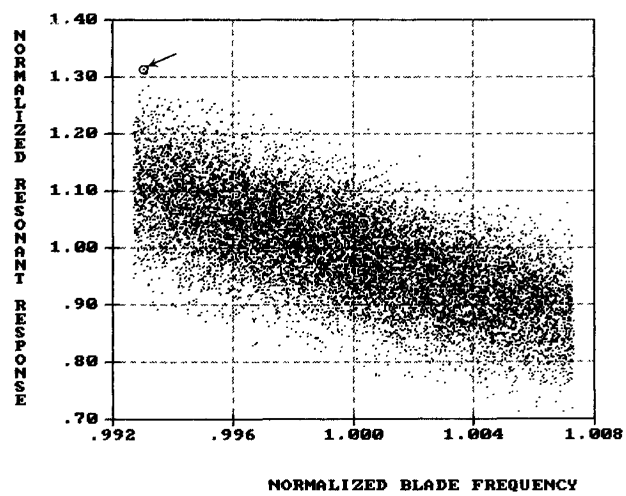

Fig. 6 Direct solution verification of the inverse solution results.

tilever frequencies were selected randomly from a normal population with a mean of $\bar{G}$ = 182 Hz and a standard deviation of σ = 0.0073$\bar{G}$. The additional constraint of all individual frequencies lying within the range of (1 ± 0.0073)$\bar{G}$ was also imposed. Response levels were computed for 2100 12-bladed disks, and the results are summarized in Fig. 6 where the maximum response of 2100 × 12 = 25,200 individual blades is shown against the normalized blade frequency. It is immediately seen that there is only one blade whose response exceeds the 30% limit with an actual value of 31.6%.

E. Relationship Between Amount of Mistuning in Blade Cantilever Frequencies and Ensuing Increase in Resonant Response Levels

The determination of the relationship between the amount of mistuning in blade cantilever frequencies and the ensuing

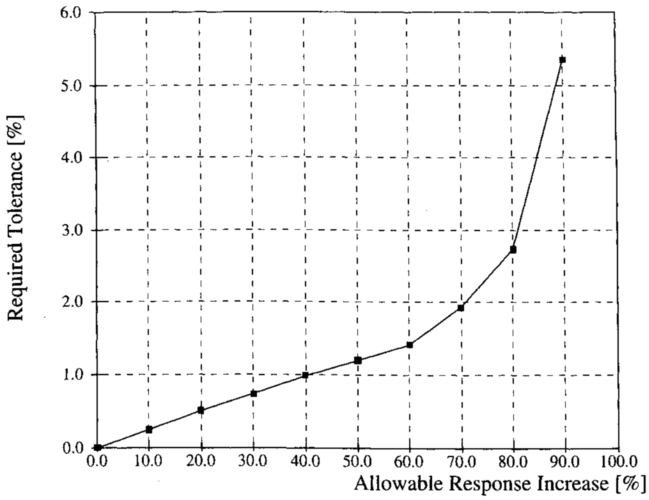

Fig. 7 Relationship between the amount of mistuning in blade cantilever frequencies and the ensuing resonant response increase.

resonant response increase is of paramount importance to turbomachinery designers because of the obvious cost implications when working to very tight manufacturing tolerances. The procedure described earlier was repeated for several values of the maximum allowable response increase, and the corresponding blade-to-blade variations were determined statistically. The results are summarized in Fig. 7 from which it can be seen that the relationship between mistuning and response increase is almost linear up to 60% response increase and an exponential trend is observed thereafter. In this latter region, to reduce the maximum response increase by 10% (say, from 90% to 80%), the amount of mistuning (and hence manufacturing tolerance) needs to be tightened by almost 100% (from $\pm 5.4\%$ to $\pm 2.8\%$). These findings seem therefore to suggest that there is a critical amount of mistuning (about $\pm 1.4\%$ for this particular case) which should be determined before attempting to reduce worsening effect of mistuning. Below that threshold, the tightening of manufacturing tolerances will have a linear effect whereas above it the reduction in response levels will be very modest.

IV. Numerical Considerations

When dealing with nonlinear inverse problems, one is often faced with several difficulties such as numerical ill conditioning, poor convergence, nonuniqueness, and nonexistence of the solution. However, it must be said at the outset that our objective here is not to focus on these numerical problems, but to record them and to find acceptable solutions with emphasis on the engineering aspects of the problem.

A. Formation and Balancing of the [S] Matrix

As can be seen from the Appendix, the nonlinear algebraic equations are numbered from 1 to $5sN$. Although the numbering of these functional relations is totally arbitrary from a mathematical standpoint, the resulting ordering of the unknowns in vector $\{\gamma\}$ is crucial from a numerical viewpoint; an inefficient numbering may result in a sparse [S] matrix, a feature which will cause the iterative solution either to diverge or to require extra iterations. Hence, the numbering of the functional relations must be made in such a way that the matrix [S] is as banded as possible.

A further problem lies in the [S] matrix itself which is ill conditioned by its nature. Indeed, the individual elements of the [S] matrix contain both response (very small numbers) and structural parameter (relatively large numbers) terms. The condition number of the matrix [S] can thus become very large, a characteristic which indicates ill conditioning. Hence, it may be necessary to balance the [S] matrix before the solution stage to overcome this ill-conditioning problem. For our bladed-disk problem, the balancing operation was performed by rescaling certain columns and rows of the [S] matrix.

Such numerical techniques improved the solution procedure remarkably: first, initially nonconverging cases started to yield solutions; second, the number of iterations required decreased; and finally, it became possible to use a pseudoinverse routine instead of a singular value decomposition one, a changeover which reduced the required CPU time by an order of magnitude.

B. Convergence and Nonuniqueness of the Solution

When solving inverse problems the iteration process may not always converge to a solution and, even if it does, that particular solution may not be unique in the sense that it can be one of the many possibilities.

To understand whether a solution is physically acceptable, let us first consider a tuned bladed-disk assembly. If the magnitudes of all blade responses are specified to be equal at one excitation frequency only, there will be more than one tuned system which will satisfy that constraint. This is illustrated in Fig. 8 where tuned blade responses are shown against excitation frequency for two different tuned bladed disks, both of which satisfy the required response level at the specified excitation frequency. Under such circumstances, the solution may converge to either of these tuned systems depending on the initial guess for vector $\{\gamma\}$. The blade cantilever frequencies of a tuned system can be found uniquely if the magnitudes of all of the tuned blade responses are specified for at least two excitation frequencies. The variation of the mistuning pattern, characterized by individual blade cantilever frequencies, with the iteration number is shown in Fig. 9, a typical example for such a case. Although all blades were assumed to have significantly different cantilever frequencies at the beginning of the iteration, convergence to the tuned blade cantilever frequency was obtained in just five iterations.

The nonexistence of a solution can also be demonstrated easily in the case of a tuned bladed-disk system: if all tuned blade response levels are specified to be greater than the tuned response, no solution can exist for a set of blade cantilever frequencies which will satisfy such response levels. As expected, the individual blade cantilever frequencies change erratically without reaching a stable value when the program is run for such a case.

The same trends are also observed in the case of mistuned assemblies in the sense that there may not be a physical system which can satisfy the assumed blade response levels. Further calculations, not reported here, indicate that it is not always possible to find a mistuning pattern, especially when the assumed response levels are not consistent. Such cases happen

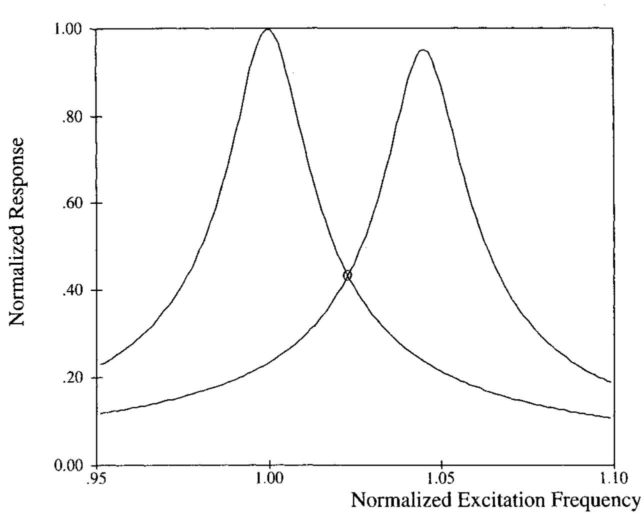

Fig. 8 Two tuned systems satisfying the specified response levels at one excitation frequency.

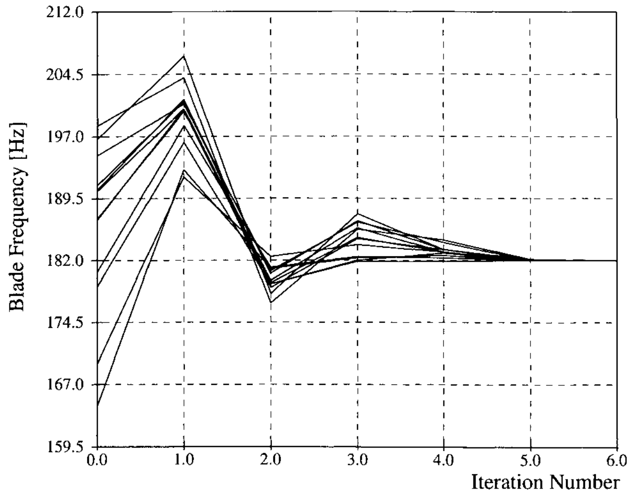

Fig. 9 Convergence rate is very fast and the solution is unique for a tuned system (blade responses are specified at more than one excitation frequency).

(1) when a large number of mistuned blades are assumed to have higher responses than the corresponding tuned resonant response or (2) when the mean value of the mistuned blade responses exceeds the corresponding tuned resonant response.

Also, the convergence rate for a mistuned system is much slower than that of the corresponding tuned system, one of the main reasons for the slower convergence is the difficulty of starting with good initial guesses for the real and imaginary parts of mistuned system's response amplitudes. As in the case of the tuned system, there may be more than one mistuned system which can satisfy the assumed response levels at a given excitation frequency. Of course, specifying the mistuned blade responses at several excitation frequencies increases the possibility of finding a unique solution; but the unavailability of data remains a major obstacle, since responses, which are related by the system's properties, cannot be specified independently from each other at several excitation frequencies.

V. Comparison of Direct and Inverse Approaches

It is clear that both the direct and the inverse approaches can be used for determining the relationship between the response increase and the required manufacturing tolerances (Fig. 7). Also, given the fact that the implementation of the direct solution is much more straightforward, the need to use the inverse approach is not immediately evident. However, as discussed subsequently, there are instances where the use of an inverse approach is more appropriate than that of its direct counterpart.

A. Design for Mistuning vs Analysis of Existing Mistuning

Mistuning analyses are conducted for one of the following two fundamentally different objectives: 1) to find the consequences of a given degree of mistuning or 2) to find the maximum allowable mistuning (or manufacturing tolerances) so that the response increase does not exceed a certain value. The first objective stems from the need to analyze an existing design which is built using blades manufactured to a known tolerance. On the other hand, the second objective originates from the basic design problem of starting with a maximum response (or stress) level and working backwards.

Direct approach is more appropriate to reach objective 1 since the amount of mistuning is known and its effects are sought. By definition, the problem is a direct one, and almost all mistuning analyses fall in this category. In total contrast, when designing a new bladed disk, the designer usually makes an assumption about the maximum allowable response (or stress) increase due to mistuning effects and determines the system parameters accordingly. The next step to be addressed is the determination of the required manufacturing tolerances so that the assumption about the maximum response increase remains valid. By definition, the problem is an inverse one, and the proposed inverse method is ideally suited for this purpose since it relates the maximum allowable response increase to manufacturing tolerances directly. On the other hand, if the designer is to use the direct method to determine the required manufacturing tolerance for a given response increase, he/she must perform a direct Monte Carlo simulation for an assumed level of mistuning and compute the corresponding maximum response increase. If the calculated maximum response (or stress) increase is different from the initial design value, the amount of mistuning must be changed and another full direct simulation must be carried out. As there is no guarantee that the response increase-mistuning relationship will be linear, many such runs may be needed before some form of interpolation can be used. Clearly, this is a trial-and-error process which can be avoided using the inverse approach.

B. Computational Cost

In the direct simulation, the mistuned blade response needs to be monitored over a wide frequency range since individual blades reach their maximum at different frequencies and the analyst has no a priori knowledge of the frequency at which the maximum response will occur. Also, several sets of calculations are needed if a mistuning range is to be considered. On the other hand, the inverse simulation requires the calculation to be performed once only at a few selected frequency points (typically between 1 and 3) regardless of the width of the frequency range.

In the example, the (IBM RS/6000 model 530) CPU cost per 100 12-bladed disks is 24 s for the direct approach for a sweep of 50 frequency points and 47 s for the inverse approach. However, with the sampling sizes being two, 100 and 500 bladed-disks, respectively, the computation for the direct method is about twice as expensive in this particular case. It should be noted that these figures are given to provide some approximate guidance for the relative computational cost and depend heavily on the sample size and the number of frequency points used. Also, they exclude wasted CPU time for nonconverging runs as it is not always possible to find an inverse solution.

Finally, one must be reminded that, for a given allowable response increase, the inverse solution yields the required manufacturing tolerances directly whereas the direct solution must use a trial-and-error path, a feature which may increase the CPU time significantly.

VI. Concluding Remarks

1) A general inverse method of solution has been developed for determining some of the parameters of a structural model when its response levels are specified at given coordinates and selected frequencies under known excitation conditions.

2) Although a number of problems arise during the numerical implementation of the method, results so far suggest that the proposed method can find a solution if one exists for the given set of structural parameters and response values. However, the identified solution may not be unique and, hence, a statistical approach might be needed in most cases.

3) The method was applied to the specific case of a 12-bladed disk to determine the mistuning patterns which would produce prescribed response levels at selected frequencies for given engine-order excitations. A statistical study was conducted over a large number of bladed disks and the solution obtained for blade mistuning patterns was found to be reliable.

4) A relationship between the manufacturing tolerances and the ensuing response increase due to mistuning was de-

termined by using several values of the allowable response increase. For the case studied, this relationship is almost linear up to 60% allowable response increase and exponential thereafter. However, the specific results reported here were obtained for a given blade-to-disk coupling ratio, engine-order excitation, and damping properties; hence it is difficult to generalize the numerical findings. In any case, the shape of this curve is of paramount importance if it is required to tighten manufacturing tolerances to reduce the forced response levels.

5) The inverse approach is ideally suited to the bladed-disk design problem where the starting point is the maximum allowable response and the requirement is the determination of the manufacturing tolerances so that the initial assumption about the maximum response level remains valid. The direct approach cannot meet the objective of finding the required manufacturing tolerances for a given allowable response increase unless the problem is solved iteratively for different tolerances, a daunting task since the response-tolerance relationship discussed in remark 4 is not expected to be linear in the general case.

Appendix: Explicit Derivation of the Sensitivity Matrix

The elements of the matrix $[S]_{5sN\times 4(s+1)N}$ are given by

$$S_{ir,ic} = \frac{\partial g_{ir}}{\partial \gamma_{ic}}$$

where ir and ic represent row (equation) and column (variable) numbers, respectively. The subscripts ir and ic are expressed in terms of frequency point ℓ and blade number j. It should be noted that if a function g_i does not depend on γ_v then the corresponding element of $[S]$ is equal to zero since $\partial g_i/\partial \gamma_v = 0$.

For $\ell = 1, 2, 3, \ldots, s$ and $j = 1, 2, 3, \ldots, N$ the nonzero elements are

$$\frac{\partial g_{5s(j-1)+5\ell-4}}{\partial \gamma_{4s(j-1)+j}} = \frac{\partial g_{5s(j-1)+5\ell-4}}{\partial G_j} = -2\zeta\omega_{\ell I}X_j(\omega_\ell) + 2G_j[{}_RX_j(\omega_\ell) - \eta_I X_j(\omega_\ell) - {}_RY_j(\omega_\ell) + \eta_I Y_j(\omega_\ell)]$$

$$\frac{\partial g_{5s(j-1)+5\ell-4}}{\partial \gamma_{4s(j-1)+4\ell+j-3}} = \frac{\partial g_{5s(j-1)+5\ell-4}}{\partial_R X_j(\omega_\ell)} = -\omega_\ell^2 + G_j^2$$

$$\frac{\partial g_{5s(j-1)+5\ell-4}}{\partial \gamma_{4s(j-1)+4\ell+j-2}} = \frac{\partial g_{5s(j-1)+5\ell-4}}{\partial_I X_j(\omega_\ell)} = -2\zeta G_j\omega_\ell - \eta G_j^2$$

$$\frac{\partial g_{5s(j-1)+5\ell-4}}{\partial \gamma_{4s(j-1)+4\ell+j-1}} = \frac{\partial g_{5s(j-1)+5\ell-4}}{\partial_R Y_j(\omega_\ell)} = -G_j^2$$

$$\frac{\partial g_{5s(j-1)+5\ell-4}}{\partial \gamma_{4s(j-1)+4\ell+j}} = \frac{\partial g_{5s(j-1)+5\ell-4}}{\partial_I Y_j(\omega_\ell)} = \eta G_j^2$$

$$\frac{\partial g_{5s(j-1)+5\ell-3}}{\partial \gamma_{4s(j-1)+j}} = \frac{\partial g_{5s(j-1)+5\ell-3}}{\partial G_j} = 2\zeta\omega_{\ell R}X_j(\omega_\ell) + 2G_j[{}_IX_j(\omega_\ell) + \eta_R X_j(\omega_\ell) - {}_IY_j(\omega_\ell) - \eta_R Y_j(\omega_\ell)]$$

$$\frac{\partial g_{5s(j-1)+5\ell-3}}{\partial \gamma_{4s(j-1)+4\ell+j-3}} = \frac{\partial g_{5s(j-1)+5\ell-3}}{\partial_R X_j(\omega_\ell)} = 2\zeta G_j\omega_\ell + \eta G_j^2$$

$$\frac{\partial g_{5s(j-1)+5\ell-3}}{\partial \gamma_{4s(j-1)+4\ell+j-2}} = \frac{\partial g_{5s(j-1)+5\ell-3}}{\partial_I X_j(\omega_\ell)} = -\omega_\ell^2 + G_j^2$$

$$\frac{\partial g_{5s(j-1)+5\ell-3}}{\partial \gamma_{4s(j-1)+4\ell+j-1}} = \frac{\partial g_{5s(j-1)+5\ell-3}}{\partial_R Y_j(\omega_\ell)} = -\eta G_j^2$$

$$\frac{\partial g_{5s(j-1)+5\ell-3}}{\partial \gamma_{4s(j-1)+4\ell+j}} = \frac{\partial g_{5s(j-1)+5\ell-3}}{\partial_I Y_j(\omega_\ell)} = -G_j^2$$

$$\frac{\partial g_{5s(j-1)+5\ell-2}}{\partial \gamma_{4s(j-1)+j}} = \frac{\partial g_{5s(j-1)+5\ell-2}}{\partial G_j} = 2G_j[{}_RY_j(\omega_\ell) - \eta_I Y_j(\omega_\ell) - {}_RX_j(\omega_\ell) + \eta_I X_j(\omega_\ell)]$$

$$\frac{\partial g_{5s(j-1)+5\ell-2}}{\partial \gamma_{4s(j-1)+4\ell+j-3}} = \frac{\partial g_{5s(j-1)+5\ell-2}}{\partial_R X_j(\omega_\ell)} = -G_j^2$$

$$\frac{\partial g_{5s(j-1)+5\ell-2}}{\partial \gamma_{4s(j-1)+4\ell+j-2}} = \frac{\partial g_{5s(j-1)+5\ell-2}}{\partial_I X_j(\omega_\ell)} = \eta G_j^2$$

$$\frac{\partial g_{5s(j-1)+5\ell-2}}{\partial \gamma_{4s(j-1)+4\ell+j-1}} = \frac{\partial g_{5s(j-1)+5\ell-2}}{\partial_R Y_j(\omega_\ell)} = -\omega_\ell^2\frac{M_d}{m} + G_j^2 + \frac{k_g}{m} + 2\frac{K_d}{m}$$

$$\frac{\partial g_{5s(j-1)+5\ell-2}}{\partial \gamma_{4s(j-1)+4\ell+j}} = \frac{\partial g_{5s(j-1)+5\ell-2}}{\partial_I Y_j(\omega_\ell)} = -\eta G_j^2$$

$$\frac{\partial g_{5s(j-1)+5\ell-2}}{\partial \gamma_{4s(j-2)+4\ell+j-2}} = \frac{\partial g_{5s(j-1)+5\ell-2}}{\partial_R Y_{j-1}(\omega_\ell)} = -\frac{K_d}{m}$$

$$\frac{\partial g_{5s(j-1)+5\ell-2}}{\partial \gamma_{4sj+4\ell+j}} = \frac{\partial g_{5s(j-1)+5\ell-2}}{\partial_R Y_{j+1}(\omega_\ell)} = -\frac{K_d}{m}$$

$$\frac{\partial g_{5s(j-1)+5\ell-1}}{\partial \gamma_{4s(j-1)+j}} = \frac{\partial g_{5s(j-1)+5\ell-1}}{\partial G_j} = 2G_j[{}_IY_j(\omega_\ell) + \eta_R Y_j(\omega_\ell) - {}_IX_j(\omega_\ell) - \eta_R X_j(\omega_\ell)]$$

$$\frac{\partial g_{5s(j-1)+5\ell-1}}{\partial \gamma_{4s(j-1)+4\ell+j-3}} = \frac{\partial g_{5s(j-1)+5\ell-1}}{\partial_R X_j(\omega_\ell)} = -\eta G_j^2$$

$$\frac{\partial g_{5s(j-1)+5\ell-1}}{\partial \gamma_{4s(j-1)+4\ell+j-2}} = \frac{\partial g_{5s(j-1)+5\ell-1}}{\partial_I X_j(\omega_\ell)} = -G_j^2$$

$$\frac{\partial g_{5s(j-1)+5\ell-1}}{\partial \gamma_{4s(j-1)+4\ell+j-1}} = \frac{\partial g_{5s(j-1)+5\ell-1}}{\partial_R Y_j(\omega_\ell)} = \eta G_j^2$$

$$\frac{\partial g_{5s(j-1)+5\ell-1}}{\partial \gamma_{4s(j-1)+4\ell+j}} = \frac{\partial g_{5s(j-1)+5\ell-1}}{\partial_I Y_j(\omega_\ell)} = -\omega_\ell^2\frac{M_d}{m} + G_j^2 + \frac{k_g}{m} + 2\frac{K_d}{m}$$

$$\frac{\partial g_{5s(j-1)+5\ell-1}}{\partial \gamma_{4s(j-2)+4\ell+j-1}} = \frac{\partial g_{5s(j-1)+5\ell-1}}{\partial_I Y_{j-1}(\omega_\ell)} = -\frac{K_d}{m}$$

$$\frac{\partial g_{5s(j-1)+5\ell-1}}{\partial \gamma_{4sj+4\ell+j+1}} = \frac{\partial g_{5s(j-1)+5\ell-1}}{\partial_I Y_{j+1}(\omega_\ell)} = -\frac{K_d}{m}$$

$$\frac{\partial g_{5s(j-1)+5\ell}}{\partial \gamma_{4s(j-1)+4\ell+j-3}} = \frac{\partial g_{5s(j-1)+5\ell}}{\partial_R X_{j(\omega\ell)}} = 2{}_RX_j(\omega_\ell)$$

$$\frac{\partial g_{5s(j-1)+5\ell}}{\partial \gamma_{4s(j-1)+4\ell+j-2}} = \frac{\partial g_{5s(j-1)+5\ell}}{\partial_I X_j(\omega_\ell)} = 2{}_IX_j(\omega_\ell)$$

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